THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Advanced Calculus I
Tutorial 11
Date: 20 June, 2025

1. Find the points on the curve $xy^2 = 54$ nearest the origin.

2. Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16+y^2/9 = 1$ with sides parallel to the coordinate axes.

3. Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.

4. (a) Show that the maximum value of $a^2b^2c^2$ on a sphere of radius r centred at the origin of a Cartesian abc-coordinate system is $(r^2/3)^3$.

   (b) Using part (a), show that for nonnegative numbers a, b, and c,

$$(abc)^{1/3} \leq \frac{a+b+c}{3}$$

   the arithmetic-geometric mean inequality for three nonnegative numbers.

1. Find the points on the curve $xy^2 = 54$ nearest the origin.

$g(x,y) = xy^2$ Geometric constraint $g(x,y) = 54$.

$f(x,y) = x^2+y^2$

$$= x^2+y^2 - \lambda xy^2 - 54\lambda$$

Then consider $F(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - 54)$

Solve for: $\vec{\nabla} F(x,y,\lambda) = 0$.

$$\Leftrightarrow \begin{cases} 0 = \frac{\partial F}{\partial x} = 2x - \lambda y^2 & (1) \\ 0 = \frac{\partial F}{\partial y} = 2y - 2\lambda xy & (2) \\ 0 = \frac{\partial F}{\partial \lambda} = -xy^2 + 54 & (3). \quad (g(x,y) = 54) \end{cases}$$

(1): $2x = \lambda y^2 \Rightarrow x = \frac{\lambda y^2}{2}$.

Sub into (2): $0 = 2y - 2\lambda\left(\frac{\lambda y^2}{2}\right)y = 2y - \lambda^2 y^3 = y(2 - \lambda^2 y^2)$

$\Rightarrow$ either $y = 0$, or $y^2 = \frac{2}{\lambda^2}$

If $y = 0$, then $x = 0$. But then $g(x,y) \neq 54$.

Sub $y^2 = \frac{2}{\lambda^2}$ into (1): $\Rightarrow 2x = \lambda y^2 = \lambda \cdot \frac{2}{\lambda^2} = \frac{2}{\lambda} \Rightarrow x = \frac{1}{\lambda}$.

Sub into (3):

$\frac{1}{\lambda} \cdot \frac{2}{\lambda^2} = 54 \Rightarrow \lambda^3 = \frac{1}{27} \Rightarrow \lambda = \frac{1}{3}$. $\Rightarrow x = 3$

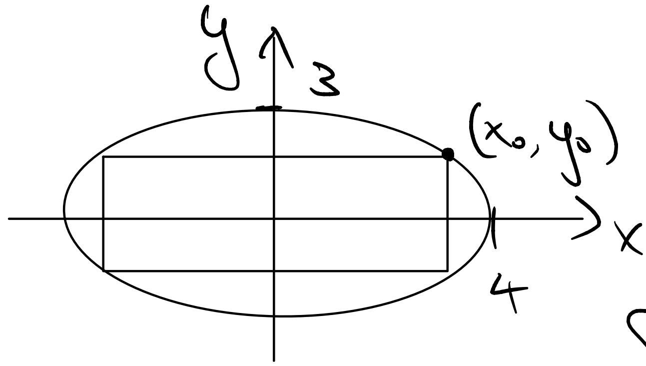
$y^2 = 18 \Rightarrow y = \pm 3\sqrt{2}$.

$f(3, 3\sqrt{2}) = 3^2 + (3\sqrt{2})^2 = 9 + 18 = 27$.

$f(3, -3\sqrt{2}) = 3^2 + (-3\sqrt{2})^2 = 27$.

So the points on $g(x,y) = 54$ nearest the origin are $(3, \pm 3\sqrt{2})$.

2. Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16+y^2/9=1$ with sides parallel to the coordinate axes.



$f(x,y) = 4xy$. = Area of inscribed rectangle.

$g(x,y) = \frac{x^2}{16} + \frac{y^2}{9} = 1$.

By Lagrange Multiplier:

$$\begin{cases} 4y = \frac{\lambda x}{8} & (1) \\ 4x = \frac{2\lambda y}{9} & (2) \\ \frac{x^2}{16} + \frac{y^2}{9} = 1. & (3) \end{cases}$$

(1), (2): $\vec{\nabla} f = \lambda \vec{\nabla} g$

(3): $g = 1$.

$(1) \Rightarrow y = \frac{\lambda x}{32}$. Sub into (2): $4\cancel{x} = \frac{2\lambda \cdot \frac{\lambda \cancel{x}}{32}}{9} \Rightarrow \lambda^2 = 4 \cdot 9 \cdot 16$

$\Rightarrow \lambda = \pm 24$.

(by (3), safe to assume $x \neq 0$).

$\lambda = 24$: (1) becomes $y = \frac{24}{32} x = \frac{3}{4} x$.

Sub into (3): $1 = \frac{x^2}{16} + \frac{(\frac{3}{4}x)^2}{9} = \frac{x^2}{8} \Rightarrow x = \pm 2\sqrt{2}$

$y = \pm \frac{3\sqrt{2}}{2}$.

$\lambda = -24$: (1) becomes $y = -\frac{3}{4} x \Rightarrow x = \pm 2\sqrt{2}$

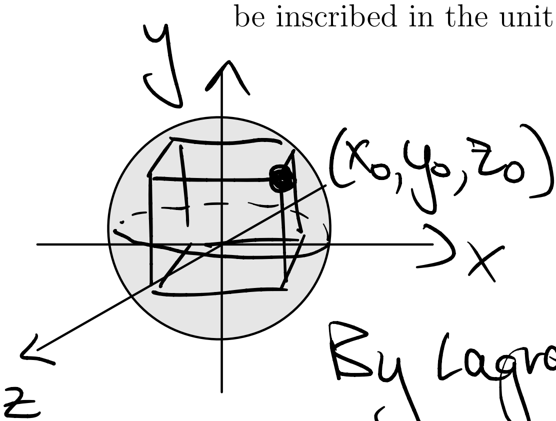
$y = \mp \frac{3\sqrt{2}}{2}$.

So area $= 2\sqrt{2} \cdot \frac{3\sqrt{2}}{2} \cdot 4 = 48$.

length $= 4\sqrt{2}$

width $= 3\sqrt{2}$. //.

3. Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.



$f(x,y,z) = 8xyz.$

$g(x,y,z) = x^2+y^2+z^2 = 1.$

By Lagrange multipliers, need to solve the system:

$$\begin{cases} 8yz = 2\lambda x & (1) \\ 8xz = 2\lambda y & (2) \\ 8xy = 2\lambda z & (3) \\ x^2+y^2+z^2=1 & (4) \end{cases}$$

Solving the system gives $\lambda = \pm\frac{4}{\sqrt{3}}$, $x=y=z=\pm\frac{1}{\sqrt{3}}$

So the dimensions of the box with maximum volume is the $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ box.

//.

4. (a) Show that the maximum value of $a^2b^2c^2$ on a sphere of radius r centred at the origin of a Cartesian abc-coordinate system is $(r^2/3)^3$.
   (b) Using part (a), show that for nonnegative numbers a, b, and c,

$$(abc)^{1/3} \leq \frac{a+b+c}{3}$$

the arithmetic-geometric mean inequality for three nonnegative numbers.

a) $f(a,b,c) = a^2b^2c^2$. $g(a,b,c) = a^2+b^2+c^2 = r^2$

By Lagrange multipliers:
$$\begin{cases} 2ab^2c^2 = 2\lambda a \quad (1) \qquad \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}. \\ 2ba^2c^2 = 2\lambda b \quad (2) \\ 2ca^2b^2 = 2\lambda c \quad (3) \\ a^2+b^2+c^2 = r^2 \quad (4). \end{cases}$$

$(1)-(2)$: $0 = 2ab^2c^2 - 2ba^2c^2 - 2\lambda a + 2\lambda b$
$= 2abc^2(b-a) + 2\lambda(b-a)$
$= (b-a)(2abc^2 + 2\lambda)$.

$\Rightarrow$ Either $a=b$, or $\lambda = -abc^2$.

When $a=b$: (3) becomes $0 = 2ca^4 - 2\lambda c = 2c(a^4 - \lambda)$

$\Rightarrow$ Either $c=0$, or $a^4 = \lambda$. $\Rightarrow a = \lambda^{\frac{1}{4}} \Rightarrow b = \lambda^{\frac{1}{4}}$

if $c=0$, then $f=0$, so reject this.

$(2)-(3)$: $0 = (c-b)(2a^2bc + 2\lambda)$

$\Rightarrow$ either $c=b$ or $\lambda = -a^2bc$

$\Rightarrow c = \lambda^{\frac{1}{4}}$.

Then in (4): $r^2 = 3(\lambda^{\frac{1}{4}})^2 = 3\lambda^{\frac{1}{2}} \Rightarrow \lambda = \frac{r^4}{9}$

$\Rightarrow a = \frac{r}{\sqrt{3}} = b = c$.

So $f\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = \left(\frac{r^2}{3}\right)^3$ //.

b) If $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ is on the sphere of radius r, then $a+b+c = r^2$

by part (a):

$$abc = f(\sqrt{a}, \sqrt{b}, \sqrt{c}) \leq \left(\frac{r^2}{3}\right)^3 = \left(\frac{a+b+c}{3}\right)^3$$

Taking cube root, get

$$(abc)^{\frac{1}{3}} \leq \frac{a+b+c}{3}$$ //.